THE FOUNDATIONS OF GEOMETRY.

THE PROBLEM OF METAGEOMETRY IS PHILOSOPHICAL.

HAVING cursorily reviewed the history of non-Euclidean geometry, which, rightly considered, is but a search for the philosophy of mathematics, I now turn to the problem itself and, in the conviction that I can offer some hints which contain its solution, I will formulate my own views in as popular language as would seem compatible with exactness. Not being a mathematician by profession I have only one excuse to offer, which is this: that I have more and more come to the conclusion that the problem is not mathematical but philosophical; and I hope that those who are competent to judge will correct me where I am mistaken.

The problem of the philosophical foundation of mathematics is closely connected with the topics of Kant's *Critique of Pure Reason.* It is the old quarrel between Empiricism and Transcendentalism. Hence our method of dealing with it will naturally be philosophical, not typically mathematical.

The proper solution can be attained only by analysing the fundamental concepts of mathematics and by tracing them to their origin. Thus alone can we know their nature as well as the field of their applicability.

We shall see that the data of mathematics are not without their premises; they are not, as the Germans say, *voraussetzungsloss*; and though mathematics is built up from nothing, the mathematician does not start with nothing. He uses mental implements, and it is they that give character to his science.

Obviously the theorem of parallel lines is one instance only of a difficulty that pops up everywhere in various forms; it is not the

disease of geometry, but a symptom of the disease. The theorem that the sum of the angles in a triangle is equal to 180 degrees; the ideas of the evenness or homaloidality of space, of the rectangularity of the square, and more remotely even the irrationality of π and of e, are all interconnected. It is not the author's intention to show their interconnection, nor to prove their interdependence. That task is the work of a mathematician. The present investigation shall be limited to the philosophical side of the problem for the sake of determining the nature of our notions of evenness, which determines both parallelism and rectangularity.

At the bottom of the difficulty there lurks the old problem of apriority, proposed by Kant and decided by him in a way which promised to give to mathematics a solid foundation in the realm of transcendental thought. And yet the transcendental method finally sent geometry away from home in search of a new domicile in the wide domain of empiricism.

Riemann, a disciple of Kant, is a transcendentalist. He starts with general notions and his arguments are deductive, leading him from the abstract down to concrete instances; but when stepping from the ethereal height of the absolute into the region of definite space-relations, he fails to find the necessary connection that characterises all *a priori* reasoning; and so he swerves into the domain of the *a posteriori* and declares that the nature of the specific features of space must be determined by experience.

The very idea seems strange to those who have been reared in traditions of the old school. An unsophisticated man, when he speaks of a straight line, means that straightness is implied thereby: and if he is told that space may be such as to render all straightest lines crooked, he will naturally be bewildered. If his metageometrical friend, with much learnedness and in sober earnest, tells him that when he sends out a ray as a straight line in a forward direction it will imperceptibly deviate and finally turn back upon his occiput, he will naturally become suspicious of the mental soundness of his company—a situation which the author can appreciate from his own experience. Having once been invited to lecture before a New Thought Club, he soon discovered to his dis-

may that all the members present, amiable though they appeared, were erratic in some way or another. They seemed quite rational on the surface, but were ready to switch off, each one on his own hobby. A man trained in Euclidean geometry must necessarily have the same impression when he hears the possibility of a fourth dimension or the curvature of space spoken of. Would we not dismiss these problems with a shrug, if there were not geniuses of first calibre who were committed to the notion that we have to suspend our judgment until we find out by experience (perhaps by experiment) what the nature of space really is?

One thing is sure, however: if there is anything wrong with metageometry, the fault lies not in its mathematical portion but must be sought for in its philosophical foundation, and it is this problem to which the following pages are devoted.

While we propose to attack the problem as a philosophical question, we hope that the solution will prove acceptable to mathematicians.

TRANSCENDENTALISM AND EMPIRICISM.

In philosophy we have the old contrast between the empiricist and transcendentalist schools. The former derive everything from experience, the latter insist that experience depends upon notions not derived from experience—called transcendental which are *a priori.* The former found their representative thinkers in Locke, Hume, and John Stuart Mill, the latter was perfected by Kant. Kant established the existence of notions of the *a priori* on a solid basis asserting their universality and necessity, but he no longer identified the *a priori* with innate ideas. He granted that much to empiricism, stating that all knowledge begins with experience and that experience rouses in our mind the *a priori* which is characteristic of mind. Mill went so far as to deny altogether necessity and universality, claiming that on some other planet 2×2 might be 5. French positivism, represented by Comte and Littré, follows the lead of Mill and thus ends in agnosticism, and the same result was reached in England by Herbert Spencer.

The way which we propose to take may be characterised as

the New Positivism. We take our stand upon the facts of experience and establish upon the systematised formal features of our experience a new conception of the *a priori*, recognising the universality and necessity of formal laws but rejecting Kant's transcendental idealism. The *a priori* is not deducible from the sensory elements of our sensations, but we trace it in the formal features of experience. It is the result of abstraction and systematisation. Thus we establish methods of dealing with experience (commonly called Pure Reason) which is possessed of universal validity, implying logical necessity.

The New Positivism is a further development of philosophic thought which combines the merits of both schools, the Transcendentalists and Empiricists, in a higher unity, discarding at the same time their aberrations. In this way it becomes possible to gain a firm basis upon the secure ground of facts, according to the principle of positivism, and yet to preserve a method established by a study of the purely formal, which will not end in nescience (the ideal of agnosticism) but justify science, and thus establishes the philosophy of science.[1]

It is from this standpoint of the philosophy of science that we propose to investigate the problem of the foundation of geometry.

THE A PRIORI AND THE PURELY FORMAL.

A priori means "beforehand," and *a posteriori* "afterwards." The bulk of our knowledge is from experience, i. e., we know things after having become acquainted with them. Our knowledge of things is *a posteriori*. If we want to know whether sugar is sweet, we must taste it. If we had not done so, and if no one had tasted it, we could not know it. But there is another kind of knowledge which we do not find out by experience, but by reflection. If I

[1] We have treated the philosophical problem of the *a priori* at full length in a discussion of Kant's *Prolegomena*. See the author's *Kant's Prolegomena*, edited in English, with an essay on Kant's Philosophy and other Supplementary Material for the study of Kant, pp. 167–240. Cf. *Fundamental Problems*, the chapters "Form and Formal Thought," pp. 26–60, and "The Old and the New Mathematics," pp. 61–73; and *Primer of Philosophy*, pp. 51–103.

want to know how much is 3×3, or $(a+b)^2$ or the angles in a regular polygon, I must compute the answer in my own mind. I need make no experiments but must perform the calculation in my own thoughts. This knowledge which is the result of pure thought is *a priori*; viz., it is generally applicable and holds good even *before* we tried it. When we begin to make experiments, we presuppose that all our *a priori* arguments, logic, arithmetic, and mathematics, will hold good.

Kant declared that the law of causation is of the same nature as arithmetical and logical truths and that, accordingly, it will have to be regarded as *a priori.* Before we make experiments, we know that every cause has its effects, and wherever there is an effect we look for its cause. Causation is not proved by, but justified through, experience.

The doctrine of the *a priori* has been much misinterpreted by Kant's followers, especially in England. Kant calls that which transcends experience in the sense that it is the condition of experience "transcendental," and comes to the conclusion that the *a priori* is transcendental. Our *a priori* notions are not derived from experience but from the conditions of experience. By experience Kant understands sense-impressions, and the sense-impressions of the outer world (which of course are *a posteriori*) are reduced to system by our transcendental notions; and thus knowledge is the product of the *a priori* and the *a posteriori.*[1]

Note here Kant's use of the word transcendental which denotes the clearest and most reliable knowledge in our economy of thought, pure logic, arithmetic, geometry. But transcendental is frequently (though erroneously) identified with "transcendent," which denotes that which transcends our knowledge and accordingly means

---

[1] A sense-impression becomes a perception by being regarded as the effect of a cause. The idea of causation is a transcendental notion. Without it experience would be impossible. An astronomer measures angles and determines the distance of the moon and of the sun. Experience furnishes the data, they are *a posteriori*; but his mathematical methods, the number system, and all arithmetical functions are *a priori.* He knows them before he collects the details of his investigation; and in so far as they are the condition without which his sense-impressions could not be transformed into knowledge, they are called transcendental.

"unknowable." Whatever is transcendental is, in Kantian terminology, never transcendent.

That much will suffice for an explanation of the historical meaning of the word transcendental. We must now explain the nature of the *a priori* and its source.

The *a priori* is identical with the purely formal which originates in our mind by abstraction. When we limit our attention to the purely relational, dropping all other features out of sight, we produce a field of abstraction in which we can construct purely formal combinations, such as numbers, or the ideas of types and species. Thus we create a world of pure thought which has the advantage of being applicable to any purely formal consideration of conditions, and we work out systems of numbers which, when counting, we can use as standards of reference for our experiences in practical life.

THE IMPORTANCE OF PURE FORM.

But if the sciences of pure form are built upon an abstraction from which all concrete features are omitted, are they not empty and useless verbiage?

Empty they are, that is true enough, but for all that they are of paramount significance, because they introduce us into the *sanctum sanctissimum* of the world, the intrinsic necessity of relations, and thus they become the key to all the riddles of the universe. That nothing (viz., the absence of any concrete materiality, implying a general anyness) from which we weave the fabric of the purely formal sciences is the realm in which Faust finds "the mothers" in whom Goethe personifies the Platonic ideas. They live in what Mephistopheles calls "the nothing," and Faust replies:

> "In deinem Nichts hoff' ich das All zu finden."
> ['Tis in thy Naught I hope to find the All.]

The formal sciences are in need of being supplemented by observation, by experience, by experiment; but while the mind of the investigator builds up purely formal systems of reference (such as numbers) and purely formal space-relations (such as geometry), the essential features of facts (of the objective world) are in their

turn, too, purely formal. That is to say: they are those features which make things such as they are. The suchness of the world is purely formal, and its suchness alone is of importance.

In studying the processes of nature we watch transformations, and all we can do is to trace the changes of form. Matter and energy are words which in their abstract significance have little value; they merely denote actuality in general, the one of being, the other of doing. What interests us most are the *forms* of matter and energy, how they change by transformation; and it is obvious that the famous law of the conservation of matter and energy is merely the reverse of the truth that causation is transformation. In its elements which in their totality are called matter and energy, existence remains the same, but the forms in which the elements combine change. The sum-total of the mass and the sum-total of the forces of the world can be neither increased nor diminished; they are the same to-day that they have always been and as they will remain forever.

All *a posteriori* experience is concrete, all *a priori* thought is abstract. The concrete is (at least πρὸς ἡμᾶς) incidental, haphazard, and individual, but the abstract is universal and can be used as a general rule under which all special cases may be subsumed.

The *a priori* is a mental construction, or, as Kant says, it is ideal, viz., it consists of the stuff that ideas are made of, it is mind-made. While we grant that the purely formal is ideal we insist that it is made in the domain of abstract thought, and its fundamental notions have been abstracted from experience by concentrating our attention upon the purely formal. It is, not directly but indirectly and ultimately, derived from experience. It is not derived from sense-experience but from a consideration of the relational (the purely formal) of experience. Thus it is a subjective reconstruction of objective features of experience and this reconstruction is made in such a way as to drop everything incidental and particular and retain only the general and essential features; and we gain the unspeakable advantage of creating rules or formulas which, though abstract and mind-made, apply to *any* case that can be classified in the same category.

Kant made the mistake of identifying the term "ideal" with "subjective," and thus his transcendental idealism was warped by the conclusion that our purely formal laws were not objective, but were imposed by our mind upon the objective world. Our mind (Kant said) is so constituted as to interpret all facts of experience in terms of form, as appearing in space and time, and as being subject to the law of cause and effect; but what things are in themselves we cannot know. We object to Kant's subjectivation of the purely formal and look upon form as an essential and inalienable feature of objective existence. The thinking subject is to other thinking subjects an object moving about in the objective world. Even when contemplating our own existence we must grant (to speak with Schopenhauer) that our bodily actualisation is our own object; i. e., we (each one of us as a real living creature) are as much objects as are all the other objects in the world. It is the objectified part of our self that in its inner experience abstracts from sense-experience the interrelational features of things, such as right and left, top and bottom, shape and figure and structure, succession, connection, etc. The formal adheres to the object and not to the subject, and every object (as soon as it develops in the natural way of evolution first into a feeling, and then into a thinking, being) will be able to build up *a priori* from the abstract notion of form in general the several systems of formal thought: arithmetic, geometry, algebra, logic, and the conceptions of time, space, and causality.

Accordingly all formal thought, although we grant its ideality, is fashioned from materials abstracted from the objective world, and it is therefore a matter of course that they are applicable to the objective world. They belong to the object and, when we thinking subjects beget them from our own minds, we are able to do so only because we are objects that live and move and have our being in the objective world.

UNIVERSALITY OF ANYNESS.

We know that facts are incidental and haphazard, and appear to be arbitrary; but we must not rest satisfied with single inci-

dents. We must gather enough single cases to make abstractions. Abstractions are products of the mind; they are subjective; but they have been derived from experience, and they are built up of elements that have objective significance.

The most important abstractions ever made by man are those that are purely relational. Everything from which the sensory element is entirely omitted, where the material is disregarded, is called "pure form," and the relational being a consideration neither of matter nor of force or energy, but of position, of shape, of size, of form, of relation, is called "the purely formal." The notion of the purely formal has been gained by abstraction, viz., by abstracting, i. e., singling out and retaining, the formal, and by thinking away, by cancelling, by omitting, by leaving out, all the features which have anything to do with the concrete sensory element of experience.

And what is the result?

We retain the formal element alone which is void of all concreteness, void of all materiality, void of all particularity. It is a mere nothing and a non-entity. It is emptiness. But one thing is left,—position or relation. Actuality is replaced by mere potentiality, viz., the possible conditions of *any* kind of being that is possessed of form.

The word "any" denotes a simple idea, and yet it contains a good deal of thought. Mathematics builds up construction to suit *any* condition. "Any" implies universality, and universality includes necessity.

In every concrete instance of an experience the subject-matter is the main thing with which we are concerned; but the purely formal aspect is after all the essential feature, because form determines the character of things, and thus the formal (on account of its anyness) is the key to their comprehension.

The rise of man above the animal is due to his ability to utilise the purely formal, as it revealed itself to him especially in types for classifying things as genera and species, in tracing transformations which present themselves as effects of causes, in shapes of measurable relations. The abstraction of the formal is made through the

instrumentality of language and the result is reason,—the faculty of abstract thought. Man can see the universal in the particular; in the single experiences he can trace the laws that are generally applicable to cases of the same class. He observes some instances and can describe them in a general formula so as to cover any other instance of the same kind.

If form were not an objective feature of the world in which we live, formal thought would never have originated and man could never have arisen. But if form is indeed (as we assume) an objective feature of all existence, it impresses itself in such a way upon living creatures that rational beings will naturally develop as soon as the organs of speech are perfected and social conditions produce a demand for communication which will result in the creation of language.

The marvellous advantages of reason dawned upon man like a revelation from on high, for he did not invent reason, he discovered it; and the sentiment that its blessings came to him from above, from heaven, from that power which sways the destiny of the whole universe, from the gods or from God, is as natural as it is true. The anthropoid did not seek reason: reason came to him and he became man. Man became man by the grace of God, by gradually imbibing the Logos that was with God in the beginning; and in the dawn of human evolution we can plainly see the landmark of mathematics, for the first grand step in the development of man as distinguished from the transitional forms of the anthropoid is the ability to count.

Man's distinctive characteristic remains, even to-day, reason, the faculty of purely formal thought; and the characteristic of reason is its anyness, implying apriority (i. e., beforehand-knowledge) which renders it possible that man can foresee events and adapt means to an end.

DIFFERENT DEGREES OF APRIORITY.

Kant has pointed out the kinship of all purely formal notions; and yet geometry cannot be derived from pure reason alone, but contains an additional element which imparts to its fundamental

conceptions an arbitrary appearance if we attempt to treat its deductions as rigidly *a priori*. Why should there be straight lines at all? Why is it possible that by quartering the circle we should have right angles with all their peculiarities? All these and similar notions can not be subsumed under a general formula from which we could derive it with logical necessity.

When dealing with lines we observe their extension in one direction, when dealing with planes we have two directions, when measuring solids we have three. Why can we not continue and construct bodies that extend in four directions? The limit set us by space as it positively presents itself to us seems arbitrary, and while transcendental truths are undeniable and obvious, the fundamental notions of geometry seem as stubborn as the facts of our concrete existence. Space, generally granted to be elbow-room for motion in all directions, after all appears to be a definite magnitude as much as a stone wall which shuts us in like a prison allowing us to proceed in three ways only and no more, and presenting no way of breaking through it. Verily, we might more easily shatter a rock that impedes our progress than pass into the fourth dimension. The boundary line is inexorable in its adamantine rigidity.

Considering all these unquestionable statements, is there not a great probability that space is a concrete fact as positive as the existence of material things, and not a mere form, not a mere potentiality of a general nature? Certainly Euclidean geometry contains such arbitrary elements as we should expect to meet in the realm of the *a posteriori*. No wonder that Gauss expressed "the desire that the Euclidean geometry should not be the true geometry," because "in that event we should have an absolute measure *a priori*."

Are we thus driven to the conclusion that our space-conception is not *a priori*; and if, indeed, it is not *a priori*, it must be *a posteriori!* What else can it be? *Tertium non datur.*

If we enter more deeply into the nature of the *a priori*, we shall learn that there are different kinds of apriority, and there is a difference between the logical *a priori* and the geometrical *a priori*.

Kant never investigated the source of the *a priori*. He discovered it in the mind and seemed satisfied with the notion that it is the nature of the mind to be possessed of time and space and the categories. He went no further. He never asked, how did mind originate?

Had Kant inquired into the origin of mind, he would have found that the *a priori* is woven into the texture of mind by the uniformities of experience. The uniformities of experience teach us the laws of form, and the purely formal involves "anyness," i. e., it is *a priori*.

Mind is the product of memory, and we may briefly describe its origin as follows:

Contact with the outer world produces impressions in sentient substance. The traces of these impressions are preserved (a condition which is called "memory") and they can be revived (which state is called "recollection"). Sense-impressions are different in kind and leave different traces, but those which are the same in kind, or similar, leave traces the forms of which are the same or similar; and sense-impressions of the same kind are registered in the traces having the same form. As a note of a definite pitch makes chords of the same pitch vibrate while it passes all others by, so new sense-impressions revive those traces only into which they fit, and thereby announce themselves as being the same in kind. Thus all sense-impressions are systematised according to their forms, and the result is an orderly arrangement of memories which is called "mind."

Thus mind develops through uniformities in sensation according to the laws of form. Whenever a new sense-perception registers itself mechanically and automatically in the trace to which it belongs, the event is tantamount to a logical judgment which declares that the object represented by the sense-impression belongs to the same class of objects which produced the memory traces with which it is registered.

If we abstract the interrelation of all memory-traces, omitting their contents, we have a pure system of genera and species, or the *a priori* idea of "classes and subclasses."

The *a priori*, though mind-made, is constructed of chips taken from the objective world, but our several *a priori* notions are by no means of the same nature and rigidity. The emptiest forms of pure thought are the categories, and the most rigid truths are the logical theorems, which can be represented by concentric circles.

If all a are A and if α is an a, then α is an A. If all dogs are quadrupeds and if all terriers are dogs, then terriers are quadrupeds. It is the most rigid kind of argument, and its statements are practically tautologies.

The case is different with causation. The class of abstract notions of which causation is an instance is much more complicated. No one doubts that every effect must have had its cause, but one of the keenest thinkers was in deep earnest when he doubted the possibility of proving this obvious statement. And Kant, seeing its kinship with geometry and algebra, accepted it as *a priori* and treated it as being on equal terms with mathematical axioms. Yet there is an additional element in the formula of causation which somehow disguises its *a priori* origin; and the reason is that it is not as rigidly *a priori* as are the norms of pure logic.

What is this additional element that somehow savors of the *a posteriori*?

If we contemplate the interrelation of genera and species and subspecies, we find that the categories with which we operate are at rest. They stand before us like a well arranged cabinet with several divisions and drawers, and these drawers have subdivisions and in these subdivisions we keep boxes. The cabinet is our *a priori* system of classification and we store in it our *a posteriori* impressions. If a thing is in box α, we seek for it in drawer a which is a subdivision of the department A.

How different is causation! While in logic everything is at rest, causation is not conceivable without motion. The norms of pure reason are static, the law of cause and effect is dynamic; and thus we have in the conception of cause and effect an additional element which is mobility.

Causation is the law of transformation. We have a definite system of interrelated items in which we observe a change of place.

The original situation and all detailed circumstances are the conditions; the motion that produces the change is the cause; the result or new arrangement of the parts of the whole system is the effect. Thus it appears that causation is only another version of the law of the conservation of matter and energy. The concrete items of the whole remain, in their constitutional elements, the same. No energy is lost; no particle of matter is annihilated; and the change that takes place is mere transformation.[1]

The law of causation is otherwise in the same predicament as the norms of logic. It can never be satisfactorily proved by experience. Experience justifies the *a priori* and verifies its tenets in single instances which prove true, but single instances can never demonstrate the universal and necessary validity of any *a priori* statement.

The logical *a priori* is rigidly *a priori*; it is the *a priori* of pure reason. But there is another kind of *a priori* which admits the use of some other abstract notion, mobility, which is part and parcel of the thinking mind. Our conception of cause and effect is just as ideal as our conception of genera and species. It is just as much mind-made as they are, and its intrinsic necessity and universal validity are the same. Its apriority cannot be doubted; but it is not rigidly *a priori*, it is only purely *a priori*.

We may classify all *a priori* notions under two headings and both are transcendental (viz., conditions of knowledge in their special fields): one is the *a priori* of being, the other of doing. The rigid *a priori* is passive anyness, the less rigid *a priori* is active anyness. Geometry belongs to the latter. Its fundamental concept of space is a product of active apriority; and thus we cannot derive its laws from pure logic alone.

The main difficulty of the parallel theorem and the straight line consists in our space-conception which is not derived from rationality in general, but results from our contemplation of motion. Our space-conception accordingly is not an idea of pure reason, but the product of pure activity.

---

[1] See the author's *Grund, Ursache und Zweck*, Dresden, 1883.

Kant felt the difference and distinguished between pure reason and pure intuition or *Anschauung*. He did not expressly say so, but his treatment suggests the idea that we ought to distinguish between two different kinds of *a priori*. Transcendental logic, and with it all common notions of Euclid, are mere applications of the law of consistency; they are (what I call in *The Primer of Philosophy*) "rigidly *a priori*." But our pure space-conception presupposes, in addition to pure reason, our own activity, the potentiality of moving about, and thus it admits another factor which cannot be derived from pure reason alone. Hence all attempts at proving the theorem on rigidly *a priori* grounds have proved failures.

SPACE AS A SPREAD OF MOTION.

Mathematicians mean to start from nothingness, so they think away everything, but they retain their own mentality. Though even their mind is stripped of all particular notions, they retain their principles of reasoning and the privilege of moving about, and from these two sources geometry can be constructed.

The idea of causation goes one step further: it admits the notions of matter and energy, emptied of all particularity, in their form of pure generalisations. It is still *a priori*, but considerably more complicated than pure reason.

The field in which the geometrician starts is pure nothingness; but we shall learn later on that nothingness is possessed of positive qualifications. We must therefore be on our guard, and we had better inquire into the nature and origin of our nothingness.

The geometrician cancels in thought all positive existence except his own mental activity and starts moving about as a mere nothing. In other words we establish by abstraction a domain of monotonous sameness, which possesses the advantage of "anyness," i. e., an absence of particularity involving universal validity. In this field of motion we proceed to produce geometrical constructions.

The geometrician's activity is pure motion, which means that it is mere progression; the ideas of a force exerted in moving and also of resistance to be overcome are absolutely excluded.

We start moving, but whither? Before us are infinite possibilities of direction. The inexhaustibleness of chances is part of the indifference as to definiteness of determining the mode of motion (be it straight or curved). Let us start at once in all possible directions which are infinite, (a proposition which, in a way, is realised by the light), and having proceeded an infinitesimal way from the starting-point A to the points B, B_1, B_2, B_3, B_4, ... B_∞; we continue to move in infinite directions at each of these stations, reaching from B the points C, C_1, C_2, C_3, C_4, ... C_∞. From B_1 we would switch off to the points $C_1^{B_1}$, $C_2^{B_1}$, $C_3^{B_1}$, $C_4^{B_1}$, ... $C_\infty^{B_1}$, etc. until we reach from B_∞ the points $C_1^{B_\infty}$, $C_2^{B_\infty}$, $C_3^{B_\infty}$, $C_4^{B_\infty}$, ... $C_\infty^{B_\infty}$, thus exhausting all the points which cluster around every B_1, B_2, B_3, B_4, ... B_∞. Thus, by moving after the fashion of the light, spreading again and again from each new point in all directions, in a medium that offers no resistance whatever, we obtain a uniform spread of light whose intensity in every point is in the inverse square of its distance from its source. Every lighted spot becomes a center of its own from which light travels on in all directions. But among these infinite directions there are rays, A, B, C . . ., i. e., lines of motion which are paths of greatest intensity. The ray of light thus ideally constructed is a representation of the straight line which, being the shortest path between the starting-point A and any other point, is the climax of directness: it is the upper limit of effectiveness and its final boundary, a *non plus ultra*. The straight line represents a climax of economy, viz., the greatest intensity on the shortest path that is reached among infinite possibilities of progression by uniformly following up all. In every ray the maximum of intensity is attained by a minimum of progression.

Our construction of motion in all directions after the fashion of light is practically pure space; but to avoid the forestalling of further implications we will call it simply the spread of motion in all directions.

The path of highest intensity in a spread of motion in all directions corresponds to the ray in an ideal conception of a spread of light, and it is equivalent to the straight line in geometry.

We purposely modify our reference to light in our construc-

tion of straight lines, for we are well aware of the fact that the notion of a ray of light as a straight line is an ideal which describes the progression of light only as it appears, not as it is. The physicist represents light as rays only when measuring its effects in reflection, etc., but when considering the nature of light, he looks upon rays as transversal oscillations of the ether. The notion of light as rays is at bottom as much an *a priori* construction as is Newton's formula of gravitation.

The construction of space as a spread of motion in all directions after the analogy of light is a summary creation of the scope of motion, and we call it "ideal space." Everything that moves about, if it develops into a thinking subject, when it forms the abstract idea of mobility, will inevitably create out of the data of its own existence the ideal "scope of motion," which is space.

When the geometrician starts to construct his figures, drawing lines and determining the position of points, etc., he tacitly presupposes the existence of a spread of motion, such as we have described. Motility is part of his equipment, and motility presupposes a field of motion, viz., space.

Space is the possibility of motion, and by ideally moving about in all possible directions the number of which is inexhaustible, we construct our notion of pure space. If we speak of space we mean this construction of our mobility. It is an *a priori* construction and is as unique as logic or arithmetic. There is but one space, and all spaces are but portions of this one construction. The problem of tridimensionality will be considered later on. Here we insist only on the objective validity of our *a priori* construction, which is the same as the objective validity of all our *a priori* constructions—of logic and arithmetic and causality, and it rests upon the same foundation. Our mathematical space omits all particularity and serves our purpose of universal application: it is founded on "anyness," and thus, within the limits of its abstraction, it holds good everywhere and under all conditions.

There is no need to find out by experience in the domain of the *a posteriori* whether pure space is curved. Anyness has no particular qualities; we create this anyness by abstraction, and it is a

matter of course that in the field of our abstraction, space will be the same throughout.

The fabric of which the purely formal is woven is an absence of concreteness. It is (so far as matter is concerned) nothing. Yet this airy nothing is a pretty tough material, just on account of its indifferent "any"-ness. Being void of particularity, it is universal; it is the same throughout, and if we proceed to build our air-castles in the domain of anyness, we shall find that considering the absence of all particularity the same construction will be the same, wherever and whenever it may be conceived.

Professor Clifford says:[1] "We *assume* that two lengths which are equal to the same length are equal to each other." But there is no "assumption" about it. The atmosphere in which our mathematical creations are begotten is sameness. Therefore the same construction is the same wherever and whenever it may be made. We consider form only; we think away all other concrete properties, both of matter and energy, mass, weight, intensity, and qualities of any kind.

UNIQUENESS OF PURE SPACE.

Our thought-forms, constructed in the realm of empty abstraction, serve as models or as systems of reference for any of our observations in the real world of sense-experience. Their anyness explains the parallelism that obtains between our models and actual experience, which was puzzling to Kant. And truly at first sight it is mystifying that a pure thought-construction can reveal to us some of the most important and deepest secrets of objective nature; but the simple solution of the mystery consists in this, that the actions of nature are determined by the same conditions of possible motions with which pure thought is confronted in its efforts to construct its models. Here as well as there we have consistency, that is to say, a thing done is uniquely determined, and, in pure thought as well as in reality, it is such as it has been made.

Our constructions are made in anyness and apply to all pos-

[1] *Loc. cit.*, p. 53.

sible instances of the kind; and thus we may as well define space as the potentiality of measuring, which presupposes moving about. Mobility granted, we can construct space as the scope of our motion in anyness. Of course we must bear in mind that our motion is in thought only and we have dropped all notions of particularity so as to leave an utter absence of force and resistance. The motor element, *qua* energy, is not taken into consideration, but we contemplate only the products of progression.

Since in the realm of pure form, thus created by abstraction, we move in a domain void of particularity, it is not an assumption (as Riemann declares in his famous inaugural dissertation), but a matter of course which follows with logical necessity, that lines are independent of position.

In actual space, position is by no means a negligible quantity. A real pyramid consisting of actual material is possessed of different qualities according to position, and the line AB, representing a path from the top of a mountain to the valley is very different from the line BA, which is the path from the valley to the top of the mountain. In Euclidean geometry $AB = BA$.

Riemann attempts to identify the mathematical space of a triple manifold with actual space and expects a proof from experience, but, properly speaking, they are radically different. In real space position is not a negligible factor, and would necessitate a fourth coördinate which has a definite relation to the plumb-line; and this fourth coördinate (which we may call a fourth dimension) suffers a constant modification of increase in inverse proportion to the square of the distance from the center of this planet of ours. It is rectilinear, yet all the plumb-lines are converging toward an inaccessible center; accordingly, they are by no means of equal value in their different parts. How different is mathematical space! It is homogeneous throughout. And it is so because we made it so by abstraction.

Pure form is a feature which is by no means a mere nonentity. Having emptied existence of all concrete actuality, and having thought away everything, we are confronted by an absolute vacancy—a zero of existence: but the zero has positive character-

istics and there is this peculiarity about the zero, that it is the mother of infinitude. The thought is so true in mathematics that it is trite. Let any number be divided by nought, the result is the infinitely great; and let nought be divided by any number, the result is the infinitely small. In thinking away everything concrete we retain with our nothingness potentiality. Potentiality is the empire of purely formal constructions, in the dim background of which lurks the phantom of infinitude.

MATHEMATICAL SPACE AND PHYSIOLOGICAL SPACE.

If we admit to our conception of space the qualities of bodies as mass, our conception of real space will become more complicated still. What we gain in concrete definiteness we lose from universality, and we can return to the general applicableness of *a priori* conditions only by dropping all concrete features and limiting our geometrical constructions to the abstract domain of pure form.

Mathematical space with its straight lines, planes, and right angles is an ideal construction. It exists in our mind only just as much as do logic and arithmetic. In the external world there are no numbers, no mathematical lines, no logarithms, no sines, tangents, nor secants. The same is true of all the formal sciences. There are no genera and species, no syllogisms, neither inductions nor deductions, running about in the world, but only concrete individuals and a concatenation of events. There are no laws that govern the motions of stars or molecules; yet there are things acting in a definite way, and their actions depend on changes in relational conditions which can be expressed in formulas. All the generalised notions of the formal sciences are mental contrivances which comprise relational features in general rules. The formulas as such are purely ideal, but the relational features which they describe are objectively real.

Thus, the space-conception of the mathematician is an ideal construction; but the ideal has objective significance. Ideal and subjective are by no means synonyms. With the help of an ideal space-conception we can acquire knowledge concerning the real

space of the objective world. Here the Newtonian law may be cited as a glaring example.

How can the subject know *a priori* anything about the object? Simply because the subject is an object moving about among objects. Mobility is a qualification of the object, and I, the thinking subject, become conscious of the general rules of motion only, because I am an object endowed with mobility. My "scope of motion" cannot be derived from the abstract idea of myself as a thinking subject, but is the product of a consideration of my mobility, generalised by omiting all particularities.

Mathematical space is *a priori* in the Kantian sense. However, it is not ready-made in our mind, it is not an innate idea, but the product of much toil and careful thought. Nor is it possible, except at a maturer age after a long development.

Physiological space is the direct and unsophisticated space-conception of our senses. It originates through experience, and is, in its way, a truer picture of actual space than mathematical space. The latter is more general, the former more concrete. In physiological space position is not indifferent, for high and low, right and left, and up and down are of great importance. Geometrically congruent figures produce (as Mach has shown) remarkably different impressions if they present themselves to the eyes in different positions.

In a geometrical plane the figures can be shoved about without suffering a change of form. If they are flopped, their inner relations remain the same, and thus they are still called congruent. Helices of opposite directions are congruent, while in actual life they would always remain mere symmetrical counterparts. So the right and the left hands considered as mere mathematical bodies are congruent, while in reality neither can take the place of the other. A glove which we may treat as a two-dimensional thing can be turned inside out, but we would need a fourth dimension to flop the hand, a three-dimensional body, into its inverted counterpart; and so long as we have no fourth dimension, the latter being a mere logical fiction, this cannot be done. Yet mathematically considered, the two hands are congruent. Why? Not because they

are actually of the same shape, but because in our mathematics position in place is excluded; the relational alone counts, and the relational is the same in both cases.

Mathematical space being an ideal construction, it is a matter of course that all mathematical problems must be settled by *a priori* operations of pure thought, and cannot be decided by external experiment or by *a posteriori* experiment.

HOMOGENEITY OF SPACE DUE TO ABSTRACTION.

When moving about, we change our place and pass by different objects. These objects too are moving; and thus our scope of motion tallies so exactly with theirs that one can be used for the computation of the other.

Space as we find it in experience is best defined as the juxtaposition of things. If there is need of distinguishing it from our ideal space-conception which is the scope of our mobility, we may call the former pure objective space, the latter pure subjective space, but, our subjective ideas being rooted in our mobility, which is a constitutional feature of our objective existence, for all practical purposes the two are the same.

But though pure space, whether its conception be established objectively or subjectively, must be accepted as the same, are we not driven to the conclusion that there are after all two different kinds of space: mathematical space, which is ideal, and physiological space, which is real? And if they are different, we must assume them to be independent of each other? What is their mutual relation?

The two spaces, the ideal construction of mathematical space and the reconstruction in our senses of the juxtaposition of things surrounding us, are different solely because they have been built up upon two different planes of abstraction; physiological space includes, and mathematical space excludes, the sensory data of juxtaposition. Physiological space admits concrete facts,—man's own upright position, gravity, perspective, etc. Mathematical space is purely formal, and to lay its foundation we have dug down to

the bed-rock of our knowledge, which is "anyness." Mathematical space is *a priori*, albeit the *a priori* of motion.

At present it is sufficient to state that the homogeneity of a mathematical space is its anyness, and its anyness is due to our construction of it in the domain of pure form, involving universality and excluding everything concrete and particular.

The idea of homogeneity in our space-conception is the tacit condition for the theorems of similarity and proportion, and also of free mobility, viz., that figures can be shoved about without suffering change, either by shrinkage or by expanse. The principle of homogeneity being admitted, we can shove figures about on any surface the curvature of which is either constant or zero. This produces either the non-Euclidean geometries of spherical, pseudo-spherical, and elliptic surfaces, or the plane geometry of Euclid—all of them *a priori* constructions made without reference to reality.

Our *a priori* constructions serve an important purpose. We use them as systems of reference. We construct *a priori* a number system, making a simple progression through a series of units which we denominate from the starting-point 0, as 1, 2, 3, 4, 5, 6, etc. These numbers are purely ideal constructions, but with their help we can count and measure and weigh the several objects of reality that confront experience; and in all cases we fall back upon our ideal number system, saying, the table has four legs; it is two and a half feet high, it weighs fifty pounds, etc. We call these modes of determination quantitative.

The element of quantitative measurement is the ideal construction of units, all of which are assumed to be discrete and equivalent. Their equivalence, as much as the homogeneity of space, is due to abstraction. In reality equivalent units do not exist any more than different parts of real space may be regarded as homogeneous. Both constructions have been made to create a domain of anyness, for the purpose of standards of reference.

EVEN BOUNDARIES AS STANDARDS OF MEASUREMENT.

Standards of reference are useful only when they are unique, and thus we cannot use any line of our spread of motion in all di-

rections, but must select one that admits of no equivocation. The only line that possesses this quality is the ray, viz., the straight line or the path of greatest intensity.

The straight line is one instance only of a whole class of similar constructions which with one name may be called "even boundaries."

Clifford, starting from objective space, constructs the plane by polishing three surfaces, *A*, *B*, and *C*, until they fit one another, which means until they are congruent.[1] His proposition leads to the same result as ours, but the essential thing is not so much (as Clifford has it) that the three planes are congruent, each to the two others, but that each plane is congruent with its own inversion. Thus, under all conditions, each one is congruent with itself. Each plane partitions the whole infinite space into two congruent halves.

Having divided space so as to make the boundary surface congruent with itself (viz., a plane), we now divide the plane (we will call it *P*) in the same way,—a process best exemplified in the folding of a sheet of paper stretched flat on the table. The crease represents a boundary congruent with itself. In contrast to curved lines, which cannot be flopped or shoved or turned without involving a change in our construction, we speak of a straight line as an even boundary.

A circle can be flopped upon itself, but it is not an even boundary congruent with itself, because the inside contents and the outside surroundings are different.

If we take a plane, represented by a piece of paper that has been evenly divided by the crease *AB*, and divide it again crosswise, say in the point *O*, by another crease *CD*, into two equal parts, we establish in the four angles round *O* a new kind of even boundary.

The bipartition results in a division of each half plane into two portions which again are congruent the one to the other; and the line in the crease *CD*, constituting, together with the first

---

[1] *Common Sense of the Exact Sciences*, Appleton & Co., p. 66.

crease AB, two angles, is (like the straight line and the plane) nothing more nor less than an even boundary construction. The right angle originates by the process of halving the straight line conceived as an angle.

Let us now consider the significance of even boundaries.

A point being a mere locus in space, has no extension whatever; it is congruent with itself on account of its want of any discriminating parts. If it rotates in any direction, it makes no difference.

There is no mystery about a point's being congruent with itself in any position. It results from our conception of a point in agreement with the abstraction we have made; but when we are confronted with lines or surfaces that are congruent with themselves we believe ourselves nonplussed; yet the mystery of a straight line is not greater than that of a point.

A line which when flopped or turned in its direction remains congruent with itself is called straight, and a surface which when flopped or turned round on itself remains congruent with itself is called plane or flat.

The straight lines and the flat surfaces are, among all possible boundaries, of special importance, for a similar reason that the abstraction of pure form is so useful. In the domain of pure form we get rid of all particularity and thus establish a norm fit for universal application. In geometry straight lines and plane surfaces are the climax of simplicity; they are void of any particularity that needs further description, or would complicate the situation, and this absence of complications in their construction is their greatest recommendation. The most important point, however, is their quality of being unique. It renders them specially available for purposes of reference.

We can construct *a priori* different surfaces that are homogeneous, yielding as many different systems of geometry. Euclidean geometry is neither more nor less true than spherical or elliptic geometry; all of them are purely formal constructions, they are *a priori*, being each one on its own premises irrefutable by experience; but plane geometry is more practical for general purposes.

The question in geometry is not, as some metageometricians would have it, "Is objective space flat or curved?" but, "Is it possible to make constructions that shall be unique so as to be serviceable as standards of reference?" The former question is due to a misconception of the nature of mathematics; the latter must be answered in the affirmative. All even boundaries are unique and can therefore be used as standards of reference.

THE STRAIGHT LINE INDISPENSABLE.

Straight lines do not exist in reality. How rough are the edges of the straightest corners, and how rugged are the straightest lines drawn with instruments of precision, if measured by the standard of mathematical straightness! And if we consider the paths of motion, be they of chemical atoms or terrestrial or celestial bodies, we shall always find them to be curves of high complexity. Nevertheless the idea of the straight line is justified by experience in so far as it helps us to analyse the complex curves into their elementary factors; each one (if we go to the end of our analysis) can be represented as a straight line. Judging from the experience we have of moving bodies, we cannot doubt that if the sun's attraction of the earth (as well as that of all other celestial bodies) could be annihilated, the earth would fly off into space in a straight line. Thus the mud on carriage wheels, when spurting off, and the pebbles that are thrown with a sling, are flying in a tangential direction which would be absolutely straight were it not for the interference of the gravity of the earth, which is constantly asserting itself and modifies the straightest line into a curve.

Our idea of a straight line is suggested to us by experience when we attempt to resolve compound forces into their constituents, but it is not traceable in experience. It is a product of our method of measurement. It is a creation of our own doings, yet it is justified by the success which attends its employment.

The great question in geometry is not, whether straight line constructions are the only possible systems of space-measurement, but what is the nature of straight lines, and planes, and right an-

gles; how does their conception originate and why are they of paramount importance in geometry.

Now the fact that the straight line (as a purely mental construction) is possible cannot be denied: we use it and that should be sufficient for all practical purposes. That we can construct curves also does not invalidate the existence of straight lines.

As to the nature of the straight line and all the other notions connected therewith, we shall always be able to determine them as concepts of boundary, either reaching the utmost limit of a certain function, be it of the highest (such as ∞) or lowest measure (such as 0); or dividing a whole into two congruent parts.

The utility of such boundary concepts becomes apparent when we are in need of standards for measurement. A boundary being the utmost limit is unique. There are innumerable curves, but there is only one kind of straight line. Accordingly, if we need a standard for measuring curves, we must naturally fall back upon the straight line and determine its curvature by its deviation from the straight line which represents a zero of curvature.

The straight line is the simplest of all boundary concepts. Hence its indispensableness.

If we measure a curvature we resolve the curve into infinitesimal pieces of straight lines, and then determine their change of direction. Thus we use the straight line as a reference in our measurement of curves. The simplest curve is the circle and its curvature is expressed by the reciprocal of the radius; but the radius is a straight line. It seems that we cannot escape straightness anywhere in geometry; for it is the simplest instrument for measuring distance. We may replace metric geometry by projective geometry, but what could projective geometricians do if they had not straight lines for their projections. Without them they would be in a strait indeed!

But suppose we renounce with Lobatchevski the conventional method of even boundary conceptions, especially straightness of line, and are satisfied with straightest lines, what is the result? He does not, at the same time, surrender either the principle of consistency or the assumption of the homogeneity of space, and thus he

builds up a geometry independent of the theorem of parallel lines, which would be applicable to two systems, the Euclidean of straight lines and the non-Euclidean of curved space. But the latter needs the straight line as much as the former and finds its natural limit in a sphere whose radius is infinite and whose curvature is zero. He can measure no spheric curvature without the radius, and after all he reaches the straight line in the limit of curvature. Yet it is noteworthy that in the Euclidean system the straight line is definite and π irrational, while in the non-Euclidean, π is a definite number according to the measure of curvature and the straight line becomes irrational.

EDITOR.

(TO BE CONCLUDED.)